Orthogonal Complement of an Orthogonal Complement

Def^n Let S be any non-empty subset of a H.S H. Then $S^\perp$ is a subspace of H. We define $(S^\perp)^\perp$ or $S^{\perp\perp}$ by $\rightarrow$

$$S^{\perp\perp} = \{x \in H \mid (x, y) = 0 \ \forall\, y \in S^\perp\}$$

Thm If S, S_1, S_2 are non-empty subsets of a H.S. H, then prove the following:

(i) $\{0\}^\perp = H$ (ii) $H^\perp = \{0\}$

(iii) $S \cap S^\perp \subseteq \{0\}$ (iv) $S_1 \subset S_2 \Rightarrow S_2^\perp \subset S_1^\perp$

(v) $S \subset S^{\perp\perp}$

Pf (i). Since $\{0\}^\perp \subset H$, $\rightarrow (1)$ $\therefore$ we shall only show that $H \subset \{0\}^\perp$.

Let $x \in H$. Since $(x, 0) = 0$, $\therefore x \in \{0\}^\perp$

$\Rightarrow H \subset \{0\}^\perp \rightarrow (2)$

$\therefore$ From (1) & (2) $\rightarrow$

$H = \{0\}^\perp$

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(ii) Let $x \in H^{\perp}$. Then,

$$\langle x, y \rangle = 0 \quad \forall\, y \in H$$

Take $y = x$, we get,

$$\langle x, x \rangle = 0 \Rightarrow \|x\|^2 = 0 \Rightarrow x = 0$$

$\therefore\ x \in H^{\perp} \Rightarrow x = 0 \Rightarrow H^{\perp} = \{0\}$

(iii) Let $x \in S \cap S^{\perp} \Rightarrow x \in S \ \&\ x \in S^{\perp}$

$x \in S^{\perp} \Rightarrow x$ is orthogonal to every vector in S.

$\Rightarrow x$ is orthogonal to x also as $x \in S$

$\Rightarrow \langle x, x \rangle = 0$

$\Rightarrow \|x\|^2 = 0 \Rightarrow x = 0$

$\therefore\ S \cap S^{\perp} \subset \{0\}$

If S is a subspace of H then $0 \in S$

Also $S^{\perp}$ is a subspace of H, $\therefore\ 0 \in S^{\perp}$

$\Rightarrow 0 \in S \cap S^{\perp}$, when S is a subspace of H.

$\Rightarrow \{0\} \subset S \cap S^{\perp}$

Hence $S \cap S^{\perp} = \{0\}$, in case when S is a subspace of H.

(iv) Let $S_1 \subset S_2$. We have,

$x \in S_2^{\perp} \Rightarrow x$ is orthogonal to every vector in S_2

Since $S_1 \subset S_2$, $\therefore\ x$ is orthogonal to every vector in S_1

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$\Rightarrow x \in S_1^\perp$

$\therefore S_2^\perp \subset S_1^\perp$

(ii) Let $x \in S \Rightarrow (x, y) = 0 \quad \forall y \in S^\perp$

$\Rightarrow x \in (S^\perp)^\perp$

$\Rightarrow S \subset (S^\perp)^\perp$

$\Rightarrow S \subset S^{\perp\perp}$

(Proved).

Th. If M is a proper closed linear subspace of a H.S. 'H' then $\exists$ a non-zero vector z_0 in H s.t. $z_0 \perp M$

Pf:- Since M is a proper subspace of H,

$\therefore \exists$ a vector $x \in H$ s.t. $x \notin M$.

Let 'd' be the distance from x to M.

Then by defn of distance, we have—

$$d = \inf \{ \|x - y\| : y \in M \}$$

Since $x \notin M$, $\therefore d > 0$.

Since M is a proper subspace of H,

$\therefore \exists$ a vector $y_0 \in M$ s.t. $\|x - y_0\| = d$.

Put $z_0 = x - y_0$. Then—

$$\|z_0\| = \|x - y_0\| = d > 0$$

$\therefore z_0$ is a non-zero vector.

Now we shall show that $z_0 \perp M$.

Let y be an arbitrary vector in M.

We shall show that $z_0 \perp y$.

For any scalar α, we have—

$$z_0 - \alpha y = x - y_0 - \alpha y = x - (y_0 + \alpha y)$$

Since M is a subspace of H and $y_0, y \in M$,

$\therefore$ $y_0 + \alpha y \in M$.

By defn of 'd', we have—

$$\| x - (y_0 + \alpha y) \| \geq d \quad \longrightarrow (1)$$

Now, $\| z_0 - \alpha y \| = \| x - (y_0 + \alpha y) \| \geq d$, from (1)

$= \| z_0 \|$

i.e. $\| z_0 - \alpha y \| \geq \| z_0 \|$

$\Rightarrow \| z_0 - \alpha y \|^2 \geq \| z_0 \|^2$

$\Rightarrow (z_0 - \alpha y, z_0 - \alpha y) - \underbrace{(z_0, z_0)}_{\| z_0 \|^2} \geq 0$

$\Rightarrow (z_0, z_0) - \bar{\alpha} (z_0, y) - \alpha (y, z_0)$
$+ \alpha \bar{\alpha} (y, y) - (z_0, z_0) \geq 0 \quad \longrightarrow (2)$

Since (2) is true for any scalar α,

$\therefore$ we take $\alpha = \beta (z_0, y)$, where 'β' is an arbitrary real number.

Then $\bar{\alpha} = \beta \overline{(z_0, y)}$

Put these values of α & $\bar{\alpha}$ in eqn (2), we get—

$-\beta \overline{(z_0, y)} (z_0, y) - \beta (z_0, y) (y, z_0) + \beta^2 (z_0, y) \overline{(z_0, y)} (y, y) \geq 0$

or $-\beta \overline{(z_0, y)} (z_0, y) - \beta (z_0, y) \overline{(z_0, y)} + \beta^2 (z_0, y) \overline{(z_0, y)} \| y \|^2 \geq 0$

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$$-2\beta|(z_0,y)|^2 + \beta^2 |(z_0,y)|^2 \|y\|^2 \geq 0$$

or $\quad \beta |(z_0,y)|^2 [\beta\|y\|^2 - 2] \geq 0 \longrightarrow (3)$

(3) is true for all real β.

Suppose $(z_0,y) \neq 0$ then if we take β as +ve, & so small that $\beta\|y\|^2 < 2$ then ~~from~~ (3), we get –

$$\underbrace{\beta}_{>0} \underbrace{|(z_0,y)|^2}_{>0} [\underbrace{\beta\|y\|^2}_{<2} - 2] < 0$$

which contradicts (3).

Hence we must have –

$$(z_0,y) = 0$$

$$\Rightarrow z_0 \perp y$$

$\therefore$ $z_0 \perp y$ is true $\forall y \in M$.

Hence $z_0 \perp M$.

(Proved)